

GERBES ON QUANTUM GROUPS

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Abstract We discuss an approach to quantum gerbes over quantum groups in terms of q -deformation of transition functions for a loop group bundle. The case of the quantum group $SU_q(2)$ is treated in some detail.

0. Introduction

In view of the recent extensive activity in the theory of gerbes and their applications in quantum field theory it is not surprising that the question arises whether there is some sort of object which could be called a 'quantum gerbe'. One should understand 'quantum' here really meaning a deformation depending on a real parameter q since a gerbe already is in a sense a quantum object: It has a well defined 'quantum number' given by its Dixmier-Douady class and this class in the physical applications is related to the chiral anomaly in quantum field theory.

A proposal for a quantum gerbe was discussed in [ABJS] in terms of deformation quantization. The underlying base space of the gerbe was replaced by noncommutative space defined as a star product algebra. A noncommutative version of a line bundle was defined, [JSW], and then a gerbe was defined as a system of local noncommutative line bundles obeying a certain cocycle condition, imitating the corresponding cocycle condition for the undeformed case.

In this paper we follow a different route to quantum gerbes. We start from the alternative definition of a gerbe (without connection and curvature) as a principal $PU(H)$ bundle over a manifold M , where $PU(H)$ is the projective unitary group of a complex Hilbert space H . The gerbe class can be nontorsion only if H is infinite-dimensional. The equivalence classes of such a bundles are parametrized by elements in $H^3(M, \mathbb{Z})$. The characteristic class is called the Dixmier-Douady class.

Concretely, the principal $PU(H)$ bundle can be obtained from a loop group bundle using a projective unitary representation of the loop group. In this paper we shall construct 'quantum loop group bundles' using transition functions in a group of matrices with entries in a quantum group.

The local approach to gerbes consists of an open cover $\{U_\lambda\}$ of M and a set of line bundles $L_{\lambda\lambda'}$ over the intersections $U_\lambda \cap U_{\lambda'}$ with given isomorphisms

$$(1) \quad L_{\lambda\lambda'} \otimes L_{\lambda'\lambda''} = L_{\lambda\lambda''}$$

on triple intersections. The relation to the global description is as follows. There is a canonical central extension

$$1 \rightarrow S^1 \rightarrow U(H) \rightarrow PU(H) \rightarrow 1$$

of the projective unitary group which gives a canonical complex line bundle L over $PU(H)$. The transition functions $\phi_{\lambda\lambda'} : U_\lambda \cap U_{\lambda'} \rightarrow PU(H)$ of the principal bundle can then be used to pull back the line bundle L over $PU(H)$ to a local line bundle $L_{\lambda\lambda'}$ over $U_\lambda \cap U_{\lambda'}$. The group structure on $U(H)$ defines an identification of $L_{\lambda\lambda'} \otimes L_{\lambda'\lambda''}$ with $L_{\lambda\lambda''}$.

In the case of the quantum group $SU_q(2)$ we shall see how the the local line bundle approach is related to the loop group bundle construction.

1. The gerbe over $SU(n)$: The undeformed case

Let us next consider the case when $M = G$ is a compact connected Lie group. Let P be the space of all smooth paths f in G starting from $f(0) = 1$ and with an arbitrary endpoint $f(1) \in G$. We also require that $f^{-1}df$ is a smooth periodic function, that is, it defines a vector potential on the unit circle S^1 . Thus we may identify P as the contractible space \mathcal{A} of smooth vector potentials on S^1 with values in the Lie algebra of G .

P is the total space of a principal bundle over G with fiber equal to the group ΩG of smooth based loops in G . Based means that $f(0) = f(1) = 1$. Since $P = \mathcal{A}$ is contractible, it is a universal bundle for ΩG . Assume that $\psi : \Omega G \rightarrow PU(H)$ is a projective representation of ΩG in the Hilbert space H . We can then define

an associated principal $PU(H)$ bundle over G in the usual way, with total space $Q = P \times_\psi PU(H)$. This is the way gerbes appear in canonical quantization in field theory, [M, CM1-2, CMM].

We shall *define* a quantum gerbe over a quantum group $SU_q(N)$ in terms of 'local transition function' with values in a loop group. This means that the transition functions are loops in a group of matrices with entries in the Hopf algebra $SU_q(n)$.

To warm up, we start from the classical case by giving explicit formulas for the transition functions. In the case of $SU(n)$ it is sufficient to select n open sets to cover the base. We choose n different points $\lambda_1, \dots, \lambda_n \neq 1$ in the unit circle in the complex plane, ordered counter clockwise, such that the product $\lambda_1 \lambda_2 \dots \lambda_n \neq 1$. For $i = 1, 2, \dots, n$ let U_i be the subset of $SU(n)$ consisting of matrices g such that λ_i is not an eigenvalue of g . This gives an open cover: If g is not in any of the sets U_i then all λ_i 's are eigenvalues of g and so $\det(g) = \lambda_1 \lambda_2 \dots \lambda_n \neq 1$, a contradiction.

An each open set U_i we have a trivialization of the bundle $P \rightarrow SU(n)$. Let $h_i(t)$ be any fixed smooth contraction of the set $S^1 \setminus \{\lambda_i\}$ with $h_i(0)$ the constant map sending $S^1 \setminus \{\lambda_i\}$ to the point 1 and $h_i(1)$ the identity map. Let $d = (d_1, \dots, d_n)$ be a diagonalization of $g \in U_i$, $g = AdA^{-1}$. Then we set

$$\psi_i(g)(t) = A(h_i(t)(d_1), \dots, h_i(t)(d_n))A^{-1}$$

and this defines a path in U_i joining g to the neutral element, i.e., we have a local section $\psi_i : U_i \rightarrow P$.

The transition function on $U_i \cap U_j$ is fixed by $\psi_j(g) = \psi_i(g)\phi_{ij}(g)$ and it takes values in ΩG .

Remark 1 Allowing the paths lie in the bigger group $GL(n, \mathbb{C})$ leads to a technical simplification in the construction of transition functions. First, for each index i we have the contraction of U_i to the point $-\lambda_i \cdot 1$ given by

$$\psi_i(g)(t) = -(1-t)\lambda_i \cdot 1 + tg.$$

Note that the matrix $\psi_i(g)(t)$ is really invertible for each $g \in U_i$ and $0 \leq t \leq 1$ by the spectral property $\text{Spec}(g) \subset S^1$. Since $GL(n, \mathbb{C})$ is connected, we can deform the constant map $\psi_i(g)(0) = -\lambda_i \cdot 1$ to the constant map $g \mapsto 1$ in an obvious way. Putting these together we obtain a homotopy connecting the constant map $g \mapsto 1$

to the identity map $g \mapsto g$ in U_i . The transition functions are defined as before, $\phi_{ij}(g) = \psi_i(g)^{-1}\psi_j(g)$, now with values in the loop group of $GL(n, \mathbb{C})$.

A third way to define the trivializations and transition functions is to use functional operator calculus. If $\lambda \in S^1$ is not in the spectrum of a unitary matrix g then the logarithm of g can be defined as

$$(2) \quad \log(g) = \int_{\gamma} \frac{\log(z)}{g - z} dz,$$

where γ is a closed loop in the complex plane $\mathbb{C} \setminus \{0\}$ which encircles every point in $S^1 \setminus \{\lambda\}$ with winding number one; the loop crosses the point λ twice such that the ray R from the origin to λ is tangential to γ at λ but otherwise γ does not cross the ray; then we can define the branch of $\log(z)$ as an analytic function with a cut along R and $(g - z)^{-1}$ is nonsingular along γ . Denoting $X = \log(g)$ we have a smooth path $\psi_i(g)(t) = e^{tX}$ connecting 1 to the point $g \in U_i$ for $\lambda = \lambda_i$.

As mentioned in the beginning, the local line bundle construction of a gerbe over $SU(n)$ is obtained by pulling back the (level k) central extension of the loop group by the transition functions. However, there is simpler construction which leads to the same gerbe.

As shown in [CM1] the basic gerbe (level $k = 1$) is constructed using a family of Dirac operators parametrized by points on G . For $G = SU(n)$ the Dirac operator D_g attached to $g \in G$ has domain consisting of smooth functions on the interval $[0, 1]$ with values in \mathbb{C}^n and boundary conditions $\psi(1) = g\psi(0)$. The Dirac operator is simply the differentiation $D_g = -i \frac{d}{dx}$ on the interval. If $g = 1$ the spectrum has multiplicity n and consists of the numbers $2\pi m, m \in \mathbb{Z}$. For arbitrary g the spectrum consists of $2\pi m + \mu_i$ where μ_1, \dots, μ_n are the eigenvalues of $-i \log(g)$.

If now λ, λ' are two distinct points on the unit circle, $\lambda = e^{i\mu}, \lambda' = e^{i\mu'}$, with $0 \leq \mu, \mu' < 2\pi$, then the eigenvalues of the Dirac operator D_g with $g \in U_{\lambda} \cap U_{\lambda'}$ in the spectral interval $]\mu, \mu'[,$ match the eigenvalues of g in the segment $]\lambda, \lambda'[,$ of the unit circle. It follows that the determinant line $DET_{\mu\mu'}$ for the Dirac operators D_g is naturally isomorphic to the top exterior power $\Lambda^{top}(E_{\lambda\lambda'})$ of the corresponding spectral subspace for g . Combining with [CM1] we get the equivalences:

Theorem. *The basic gerbe over $G = SU(n)$, corresponding to the Dixmier-Douady class given by the generator of $H^3(SU(n), \mathbb{Z})$, can be given in three equivalent ways:*

- (1) *as the $PU(H)$ bundle over G obtained as an associated bundle to the univer-*

sal bundle $\mathcal{A} \rightarrow G$ through the basic representation of the affine Kac-Moody group based on G

- (2) as the local system of complex line bundles defined as above by Dirac operators on the unit interval, parametrized by boundary conditions $g \in G$
- (3) as the system of local line bundles formed from top exterior powers of the spectral subspaces of $g \in G$ corresponding to open segments of the unit circle.

Let us consider the case $G = SU(2)$ in more detail. It is sufficient to choose a cover consisting of two open sets U_{\pm} consisting of points $g \neq \pm 1$. The overlap is homotopic to the equator S^2 in $SU(2) = S^3$. The basic transition function for a loop group bundle on $SU(2)$ is then

$$(3) \quad \phi(x)(t) = \cos(2\pi t) + ix \sin(2\pi t) \text{ for } 0 \leq t \leq \frac{1}{2}$$

and $\phi(x)(t) = \text{diag}(e^{2\pi it}, e^{-2\pi it})$ for $\frac{1}{2} \leq t \leq 1$. Here $x = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix}$. Actually, to make the function smooth at $t = 1/2$ and the end points one should replace the variable t by $f(t)$ where f is any smooth monotonous function on the interval $[0, 1]$ such that $f(0) = 0, f(1) = 1$, and all the derivatives vanish at the points $t = 0, 1/2, 1$.

2. The gerbe over the quantum group $SU_q(n)$

We assume $0 < q < \infty$ and denote by $SU_q(n)$ the standard Hopf algebra of the quantum special unitary group. It is given in terms of generators and relations as follows. The generators are elements g_{ij} indexed by $i, j = 1, 2, \dots, n$ with defining relations

$$(4) \quad \begin{aligned} g_{im}g_{ik} &= qg_{ik}g_{im}, \quad g_{jm}g_{im} = qg_{im}g_{jm} \\ g_{im}g_{jk} &= g_{jk}g_{im}, \quad g_{ik}g_{jm} - g_{jm}g_{ik} = (q^{-1} - q)g_{im}g_{jk} \end{aligned}$$

for $i < j$ and $k < m$. In addition, the quantum determinant

$$(5) \quad \det_q = \sum_{\sigma \in S_n} (-q)^{-\ell(\sigma)} g_{1\sigma(1)} g_{2\sigma(2)} \cdots g_{n\sigma(n)}$$

is identified as the unit element in the algebra.

The counit is defined by $\epsilon(g_{ij}) = \delta_{ij}$ and the antipode is defined as

$$(6) \quad S(g_{ij}) = (-1)^{i+j} X_{ji},$$

where X_{ij} is the $(n-1) \times (n-1)$ quantum minor of the matrix (g_{ij}) , i.e., the quantum determinant of the submatrix obtained by deleting the i :th row and j :th column from (g_{ij}) . The star algebra structure is given as

$$(7) \quad g_{ij}^* = S(g_{ji}).$$

In particular, by the definition of an antipode, $\sum_{(x)} x' S(x'') = \epsilon(x) \cdot 1$ where $\Delta(x) = \sum_{(x)} x' \otimes x''$, and so

$$(8) \quad g_{ij} g_{ik}^* = \delta_{jk} \cdot 1.$$

The coproduct Δ is defined by matrix multiplication,

$$\Delta(g_{ij}) = \sum_k g_{ik} \otimes g_{kj}.$$

For more details, see [KS].

In the case $n = 2$ the standard notation is $a = g_{11}, b = g_{12}, c = g_{21}, d = g_{22}$ with $b^* = -qc, a^* = d$. The determinant condition is $ad - qbc = 1$ and the commutation relations are

$$(9) \quad ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \quad ad - da = (q - q^{-1})bc.$$

We shall now define a 'quantum gerbe' over the quantum group $SU_q(2)$ as a quantum projective bundle over $SU_q(2)$. This is achieved by giving the transition function on the 'equator' of $SU_q(2)$. The quantum equator is defined as a quotient algebra of $SU_q(2)$, [HMS]. Let $I \subset SU_q(2)$ be the 2-sided ideal generated by the single element $b - b^* = b + qc$. Then one can show that the standard quantum sphere S_q^2 is isomorphic to $SU_q(2)/I$, [HMS]. Explicitly, the generators of S_q^2 are $K = K^*$ and L with the defining relations

$$(10) \quad LL^* + q^2 K^2 = 1, \quad L^* L + K^2 = 1, \quad LK = qKL.$$

One can check that $K = c + I, L = a + I$ in $SU_q(2)/I$ indeed satisfy the relations above. In the classical case, $q = 1$, going to the quotient means that we set one of the coordinates, namely $\text{Im}(b)$, equal to zero reducing the 3-sphere to the equatorial 2-sphere.

The transition function is defined as in the undeformed case: Define the quantum 2×2 matrix

$$(11) \quad x = \begin{pmatrix} qK & L \\ L^* & -K \end{pmatrix}.$$

It is easy to check from the defining relations that $x^* = x$ (combination of matrix transposition and star operation on S_q^2) and that $x^2 = 1$. It follows that $\cos(2\pi t) + ix \sin(2\pi t)$ ($0 \leq t \leq 1/2$) is unitary and in combination with the path $t \mapsto \text{diag}(e^{2\pi it}, e^{-2\pi it})$ ($1/2 \leq t \leq 1$) defines a loop in unitary 2×2 matrices with coefficients in the algebra S_q^2 .

In the classical undeformed case the transition function can be extended from the equator to the open set $U_{+-} = SU(2) \setminus \{\pm\sigma\}$ where $\sigma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. First one extends the function x to U_{+-} by writing

$$(12) \quad x = f^{-1} \begin{pmatrix} \frac{1}{2}(b + b^*) & a^* \\ a & -\frac{1}{2}(b + b^*) \end{pmatrix},$$

where $f = [1 + \frac{1}{2}(b - b^*)^2]^{1/2}$. This function is singular at the points where $\text{Im}(b) = \pm i$, that is, at opposite poles of $S^3 = SU(2)$. It satisfies $x^* = x = x^2$ and its restriction to the equator defined by $\text{Im}(b) = 0$ is equal to $\begin{pmatrix} b & d \\ a & -b \end{pmatrix}$. We may also view x as a matrix with entries in an algebraic extension of the commutative algebra of functions on $SU(2)$ by a single element f satisfying the defining relation $f^2 = 1 + \frac{1}{4}(b - b^*)^2$. That is, we add certain functions with singularities at the poles to the algebra generated by the elements a, b, c, d .

The generalization of the above formula to the q -deformed case is given as

$$(13) \quad x = \begin{pmatrix} \frac{1}{2q}(b + b^*)f_q^{-1} & a^*f^{-1} \\ f^{-1}a & -\frac{1}{2}(b + b^*)f^{-1} \end{pmatrix}$$

where f is same as before and $f_q = [1 + \frac{1}{4q^2}(b - b^*)^2]^{1/2}$. This means that the matrix x has entries in an algebraic extension of $SU_q(2)$ by the elements f, f_q satisfying the relations $f^2 = 1 + \frac{1}{4}(b - b^*)^2$ and likewise for f_q .

As in the undeformed case, we can think of the quantum gerbe coming from a K-theory class defined by a family of Dirac type operators. The family of operators is again given by the differentiation $D = -i\frac{d}{dt}$ but now acting on 2-component spinors with coefficients in the quantum group $SU_q(2)$. The family of boundary conditions is

$$\psi(1) = g\psi(0)$$

where g is the quantum matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. So in a sense the K-theory class in $K^1(SU_q(2))$ is tautological: It is the unitary 2×2 matrix g given by the 'coordinates' of the quantum group. Note that the K-theory of $SU_q(2)$ when $q > 0$ is the same as for $q = 1$, $K^1(SU_q(2)) = \mathbb{Z} = K^0(SU_q(2))$, [MNW2].

The quantum line bundle on the equator S_q^2 is defined as the projection $P = \frac{1}{2}(1 + x)$ with coefficients in S_q^2 . The property $P^2 = P$ follows from $x^2 = 1$. Of course the case of $SU_q(2)$ is very simple since there is only one 'overlap' S_q^2 and we do not need to bother about the generalization of (1) to the quantum group case.

Remark 2 The rank one projectors in the q deformed case define only right S_q^2 modules and not bimodules since the left multiplication by the elements in the algebra S_q^2 does not commute with the multiplication by P on $S_q^2 \oplus S_q^2$. For this reason the tensor products of line bundles over S_q^2 are not canonically defined.

Since there is no underlying smooth manifold, the Dixmier-Douady class cannot be defined in terms of de Rham forms as in the undeformed case. Instead, one can use a cyclic cocycle c_3 to compute the quantum invariant of the gerbe by pairing c_3 with the K-theory class of the unitary matrix g . This was in fact already done in [Co] (in the case $0 < q < 1$) and I will not repeat (the rather complicated) calculations here.

Next we want to generalize the construction to $G = SU_q(n)$. This is achieved by a generalization of the method in Remark 1. Let $\lambda \in S^1$. Define an algebraic extension $SU_q(n)_\lambda$ of $SU_q(n)$ by requiring that the equation $(g - \lambda)h = 1$ has a solution h as a $n \times n$ matrix with entries in the extended algebra. In the classical case this means that we allow singularities for $\det(g - \lambda)$, i.e., we restrict the domain of functions on $SU(n)$ to the open subset specified by $\lambda \notin \text{Spec}(g)$.

Actually, we need to consider the algebra $SU_q(n)$ as a $*$ -algebra completion of the algebra defined by the relations (4). Then $g - t\lambda$ has an inverse for any real

number $t \neq 1$ since g is a unitary matrix. We have then a homotopy

$$(14) \quad \psi_t(g, \lambda) = -(1-t)\lambda + tg$$

connecting $\psi_0(g, \lambda) = -\lambda$ to the identity map $g \mapsto g = \psi_1(g, \lambda)$. The homotopy is defined in the space of nonsingular matrices with entries in $SU_q(n)_\lambda$. We extend this homotopy to a path connecting the constant map $g \mapsto 1$ to the identity $g \mapsto g$ in an obvious way, along the path $\psi_t = (-\lambda)^t$, $0 \leq t \leq 1$ connecting the neutral element to λ .

The 'transition functions' are defined as in the undeformed case, $\phi_{\lambda\lambda'}(g)(t) = \psi_t(g, \lambda)\psi_t(g, \lambda')^{-1}$. The transition functions are loops of $n \times n$ matrices with coefficients in the extension $SU_q(n)_{\lambda\lambda'}$ defined by the inverses $(g - \lambda)^{-1}, (g - \lambda')^{-1}$.

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